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Existence and Global Higher Integrability of Quasiminimizers among Minimizing Sequences of Variational Integrals

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ABSTRACT

Quasiminimizers can be viewed as the perturbations of minimizers of variational integrals. We first establish the existence of good minimizing sequences of non-trivial variational integrals containing quasiminimizers of an inhomogeneous p-Dirichlet integral. Employing the concept of variational capacity, we show that the gradients of these quasiminimizers possess global higher integrability.

Keywords: Quasiminimizers, integrability, minimizing sequences, p-Dirichlet integral, variational problems.

1. Introduction

The notion of quasiminimizer can be viewed as perturbations of minimizers of variational integrals. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open set,

$p \in (1, \infty)$, $Q \in [1, \infty)$ and $a \in L^1(\Omega)^+$. We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ is a Q-quasiminimizer of the inhomogeneous p-Dirichlet integral if the inequality

$$\int_K \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x \le Q \int_K \big(|Dv(x)|^p + a(x)\big)\,\mathrm{d}x \tag{1}$$

holds for all $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ with $K = \mathrm{supp}(v - u) \Subset \Omega$. We consider cubical or sphecial Q-quasiminimizers if the integrals in the definition above are taken over cubes or balls of $\mathbb{R}^n$, respectively.

If $Q = 1$, then u is a standard minimizer of the inhomogeneous p-Dirichlet integral. When $N = 1$ so that u and v are real-valued, we consider Q-superquasiminimi-zer in which the quasiminimality inequality (1) holds for all $v \ge u$ and Q-subquasimi-nimizer for $v \le u$.

Originally introduced by Giaquinta and Giusti (1984), the theory of Q-quasimini-mizer is used as a tool for unified treatment of variational integrals, elliptic equations and quasiregular mappings. Due to its wide range of applications that lead to a broad and flexible class of maps under general circumstances, it has since been extended to metric space (Björn (2002) and Kinnunen and Shanmugalingam (2001)) and potential theory (Kinnunen and Martio (2003)).

Some took interest in studying scalar-valued quasiminimizers; for example one-dimensional quasiminimizers (Giaquinta and Giusti (1984), Martio (2009) and Martio and Sbordone (2007)), radial or power-type quasiminimizers (Björn and Björn (2011)) and local quasiminimizers (Björn and Björn (2011) and Kinnunen and Martio (2003)).

Apart from that, the strength of Q-quasiminimizer also lies in its regularity properties, which include in the scalar case, Hölder continuity, weak maximum principle and Harnack inequality, and for the vectorial case higher integrability in the interior. This is in response to Hilbert's 19th problem from his famous list (see Hilbert (1902)), since a minimizer of a variational integral is a function in a Sobolev space and hence will not be continuous in general.

This paper continues the study of Q-quasiminimizers in the direction as above; first to show that the notion can be extended to the variational problem

$$\inf\Big\{\mathcal{F}(u, \Omega) \colon u - g \in W^{1,p}(\Omega; \mathbb{R}^N)\Big\},$$

where functional $\mathcal{F}(u, \Omega) = \int_\Omega F\big(x, u(x), Du(x)\big)\,\mathrm{d}x$, $g \in W^{1,p}(\Omega; \mathbb{R}^N)$ and integrand $F \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ satisfies some coercivity and growth conditions. It turns out that if there exists a good minimizing sequence for the variational integral, then we can always find another minimizing sequence consisting of Q-quasiminimizers of the inhomogeneous p-Dirichlet integral. This is then followed by the regularity property of these Q-quasiminimizers, namely global higher integrability of the derivatives.

This paper is organized as follows: In Section 2, we discuss about the existence of Q-quasiminimizers within good minimizing sequences of the variational problem. We then continue our discussion on global higher integrability in Section 3, but this will only be done after a brief discussion on variational capacity and higher integrability in the interior. Lastly, we give a concluding remark in Section 4.

2. Existence of Quasiminimizers

This section further establishes the extension of Q-quasiminimizers to variational problems. Given that a functional satisfies growth (2) and coercivity (3) conditions (see Theorem 3.1 below) and the corresponding variational problem has a minimizing sequence, we show that there exists another minimizing sequence of the variational problem consisting of Q-quasiminimizers of the inhomogeneous p-Dirichlet integral.

The idea of finding good minimizing sequences in the manner described above goes back to at least Marcellini and Sbordone (1980). Our construction of such sequences follows from that of Yan and Zhou (1997) but in addition, generalized to local functionals of p-growth. A functional $\mathcal{F}\colon W^{1,p}(\Omega;\mathbb{R}^N)\times\mathcal{O}(\Omega)\to\mathbb{R}$ is said to be local on $\mathcal{O}(\Omega)$ if for $u,v\in W^{1,p}(\Omega;\mathbb{R}^N)$

$$\mathcal{F}(u,\omega)=\mathcal{F}(v,\omega) \text{ whenever } u=v \text{ a.e. on } \omega\in\mathcal{O}(\Omega),$$

where $\mathcal{O}(\Omega)$ denotes the family of all open subsets of Ω.

Theorem 2.1. *Let $\mathcal{F}\colon W^{1,p}(\Omega;\mathbb{R}^N)\times\mathcal{O}(\Omega)\to\mathbb{R}$ be a local functional that is $W^{1,1}$ lower semicontinuous on $\mathcal{O}(\Omega)$ and assume that*

$$|\mathcal{F}(u,\omega)|\le c_1\int_\omega (|Du|^p+1)\,\mathrm{d}x \tag{2}$$

for all $u\in W^{1,p}(\omega;\mathbb{R}^N)$ and for all $\omega\in\mathcal{O}(\Omega)$ where $c_1<\infty$ and $p>1$ are constants. Assume furthermore that

$$\mathcal{F}(u,\omega)+c_2\int_\omega (|Dg|^p+1)\,\mathrm{d}x\ge c_3\int_\omega |Du|^p\,\mathrm{d}x \tag{3}$$

for all $g \in W^{1,p}(\omega;\mathbb{R}^N)$, $u \in W^{1,p}_g(\omega;\mathbb{R}^N)$ *and* $\omega \in \mathcal{O}(\Omega)$ *where constants* $c_2 < \infty$ *and* $c_3 > 0$. *Then there exists a constant* $Q = Q(c_1, c_2, c_3, p) < \infty$ *such that if* $\{u_j\}$ *is a minimizing sequence for the variational problem*

$$\inf\left\{\mathcal{F}(u,\Omega):\, u \in W^{1,p}_g(\Omega;\mathbb{R}^N)\right\}, \tag{4}$$

then there exists another minimizing sequence $\{v_j\}$ *for* (4) *consisting of* Q*-quasimini-mizers of the inhomogeneous* p*-Dirichlet integral*

$$\int_\Omega (|Du|^p + 1)\,\mathrm{d}x \tag{5}$$

and such that

$$\int_\Omega |Dv_j - Du_j|\,\mathrm{d}x \to 0. \tag{6}$$

Proof. The main essence of this proof is based on the proof of Theorem 6.3 in Giaquinta and Giusti (1984). Let

$$m = \inf\left\{\mathcal{F}(u,\Omega):\, u \in W^{1,p}_g(\Omega;\mathbb{R}^N)\right\}.$$

It is clear that $m > -\infty$ by (3). Let $\varepsilon_j = \mathcal{F}(u_j,\Omega) - m$. For $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, we define a complete metric space (X,d) by

$$X = W^{1,1}_g(\Omega;\mathbb{R}^N)$$

$$d = d(w,v) = \int_\Omega |Dw - Dv|\,\mathrm{d}x.$$

By assumption $\mathcal{F}(v,\Omega)$ is lower semicontinuous on (X,d). Hence Ekeland's variational principle, see Ekeland (1974), yields $v_j \in W^{1,p}_g(\Omega;\mathbb{R}^N)$ such that

$$\begin{cases} \mathcal{F}(v_j,\Omega) \le \mathcal{F}(u_j,\Omega), \\ \int_\Omega |Du_j - Dv_j|\,\mathrm{d}x \le \varepsilon_j, \\ \mathcal{F}(v_j,\Omega) \le \mathcal{F}(u,\Omega) + \varepsilon_j \int_\Omega |Du - Dv_j|\,\mathrm{d}x, \end{cases} \tag{7}$$

for all $u \in W^{1,p}_g(\Omega;\mathbb{R}^N)$. Since $\varepsilon_j > 0$ can be chosen to be arbitrarily small, we obtain (6).

To prove that v_j is quasiminimizing, fix j and let $\omega \in \mathcal{O}(\Omega)$. Then, by (3), we have for all $v_j \in W^{1,p}_u(\omega,\mathbb{R}^N)$

$$c_3 \int_\omega |Dv_j|^p\,\mathrm{d}x \le \mathcal{F}(v_j,\omega) + c_2 \int_\omega (|Du|^p + 1)\,\mathrm{d}x.$$

If we define

$$\tilde{u} = \begin{cases} u & \text{in} \quad \omega \\ v_j & \text{in} \quad \Omega \setminus \omega \end{cases}$$

with $u = v_j$ on $\partial\omega$, then $\tilde{u} \in W_g^{1,p}(\Omega; \mathbb{R}^N)$. Taking $\tilde{u}$ into the last inequality of (7), we have

$$\mathcal{F}(v_j, \omega) \leq \mathcal{F}(u, \omega) + \varepsilon_j \int_\omega |Du - Dv_j| \, \mathrm{d}x$$

and subsequently

$$c_3 \int_\omega |Dv_j|^p \, \mathrm{d}x \leq \mathcal{F}(u, \omega) + \varepsilon_j \int_\omega |Du - Dv_j| \, \mathrm{d}x + c_2 \int_\omega (|Du|^p + 1) \, \mathrm{d}x.$$

By growth condition (2) and triangle inequality, we have

$$c_3 \int_\omega |Dv_j|^p \, \mathrm{d}x \leq (c_1 + c_2) \int_\omega (|Du|^p + 1) \, \mathrm{d}x + \frac{c_3}{2} \int_\omega (|Du| + |Dv_j|) \, \mathrm{d}x,$$

where $c_3 > 2\varepsilon_j$. Invoking Young's inequality on the last term of the right hand side and performing simple algebra gives us

$$c_3 \int_\omega (|Dv_j|^p + 1) \, \mathrm{d}x \leq \left(c_1 + c_2 + \frac{3c_3}{2} \right) \int_\omega (|Du|^p + 1) \, \mathrm{d}x + \frac{c_3}{2} \int_\omega (|Dv_j|^p + 1) \, \mathrm{d}x.$$

In the end, we arrive at

$$\int_\omega (|Dv_j|^p + 1) \, \mathrm{d}x \leq Q \int_\omega (|Du|^p + 1) \, \mathrm{d}x.$$

where $Q = (2c_1 + 2c_2 + 3c_3)/c_3$. □

3. Global Higher Integrability of Quasiminimizers

It can be shown that these Q-quasiminimizers of the inhomogeneous p-Dirichlet integral possess self-improving property, by means of Gehring's lemma (Gehring (1973)), which leads to regularity in the interior. If the set is of p-capacity zero (or removability of singularities), then global higher integrability is also attainable. Note that, in the paper of Giaquinta and Giusti (1984), there are coercivity and growth conditions imposed on the integrand F.

Following the idea of Kilpeläinen and Koskela (1994), we show that global higher integrability can be attained even without any coercivity and growth conditions, provided that the complement of Ω is uniformly p-thick.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set such that Ω^c is uniformly p-thick (no restriction when $p > n$) and $c_0 > 0$ is the uniform p-thickness constant given in Definition 3.2 below. Let $g \in W^{1,\tilde{p}}(\Omega;\mathbb{R}^N)$ for some $\tilde{p} > p$. If $u \in W^{1,p}_g(\Omega;\mathbb{R}^N)$ is a Q-quasiminimizer of the inhomogeneous p-Dirichlet integral*

$$\int_\Omega (|Dv|^p + 1)\,\mathrm{d}x,$$

then there exists a constant $\delta_0 = \delta_0(n, N, p, Q, c_0) \in (0, \tilde{p} - p]$ such that $u \in W^{1,p+\delta}(\Omega;\mathbb{R}^N)$ whenever $\delta \in (0, \delta_0)$. In particular, $|Du| \in L^{p+\delta}(\Omega)$ for all $\delta \in (0, \delta_0)$ and

$$\left(\int_\Omega |Du|^{p+\delta}\,\mathrm{d}x\right)^{\frac{1}{p+\delta}} \le c\left(\int_\Omega |Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}} + c\left(\int_\Omega |Dg|^{p+\delta}\,\mathrm{d}x\right)^{\frac{1}{p+\delta}} + c$$

where constant $c = c(n, N, p, Q, c_0) \in (0, \infty)$.

3.1 Variational capacity

Uniform p-thickness is a weaker form of regularity of the boundary of a set of Ω as compared to Lipschitz boundary, but is nevertheless sufficient for proving global higher integrability results (see Heinonen et al. (1993)). The definition of uniform p-thickness will follow after our formal introduction of variational p-capacity (or p-capacity). We will also include a series of results which will be used in the proof of Theorem 3.1.

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The p-capacity of a compact set $K \subset \Omega$ is defined as*

$$\mathrm{cap}_p(K;\Omega) := \inf\left\{\int_\Omega |Du|^p \mathrm{d}x \colon u \in C_0^\infty(\Omega),\ u = 1 \text{ on } K\right\}$$

and for an arbitrary set $A \subset \Omega$,

$$\mathrm{cap}_p(A;\Omega) := \inf_{\substack{A\subset E\subset\Omega \\ E \text{ open}}}\ \sup_{\substack{K\subset E \\ K \text{ compact}}} \mathrm{cap}_p(K;\Omega).$$

Definition 3.2. *Let $p \in (1, \infty)$. A set Ω is said to be uniformly p-thick if there exist positive constants c_0 and R such that*

$$\mathrm{cap}_p\left(\Omega \cap \bar{B}(x, r); B(x, 2r)\right) \ge c_0\ \mathrm{cap}_p\left(\bar{B}(x, r); B(x, 2r)\right)$$

whenever $x \in \Omega$ and $r \in (0, R)$.

It follows immediately that from the definition of p-capacity that every non-empty set is uniformly p-thick when $p > n$, leaving us with the case when $p \leq n$. A uniformly p-thick set possesses a self-improving property:

Theorem 3.2 (Lewis (1988)). *Let $p \in (1, n]$. If a set Ω is uniformly p-thick, then there exists $q \in (1, p)$ such that Ω is uniformly q-thick.*

The p-capacity replaces the Lebesgue measure in Lusin's type theorems for $W^{1,p}$ Sobolev functions. This leads to the notion of p-quasicontinuity, which in turn gives us a precise representative of Sobolev functions and subsequently a version of Sobolev inequality. The relevant notion here is highlighted in the next definition.

Definition 3.3. *A function u is p-quasicontinuous if for each $\varepsilon > 0$, there exists an open set $\omega \subset \Omega$ such that $\operatorname{cap}_p(\omega; \Omega) \leq \varepsilon$ and the restriction of u to $\Omega \setminus \omega$ is continuous.*

Theorem 3.3 (Ziemer (1989)). *Let $\Omega \subset \mathbb{R}^n$ be open. For each $u \in W^{1,p}(\Omega)$, there is a p-quasicontinuous representative such that*

$$\lim_{r \to 0} \fint_{B(x,r)} u(y)\, \mathrm{d}y = u(x)$$

for all $x \in \Omega$ except on a set E of p-capacity zero.

Lemma 3.1 (**Sobolev inequality**, Maz'ya (1985)). *Suppose that $q \in (1, \infty)$ and that u is a q-quasicontinuous function in $W^{1,q}(B)$, where B is a ball. Let $N(u) = \{x \in B \colon u(x) = 0\}$. Then*

$$\left(\fint_B |u|^{\kappa q}\, \mathrm{d}x \right)^{1/\kappa q} \leq c \left(\frac{1}{\operatorname{cap}_q(N(u); 2B)} \int_B |Du|^q\, \mathrm{d}x \right)^{1/q}$$

where $c = c(n, q) > 0$ and

$$\kappa = \begin{cases} \frac{n}{n-q} & \text{if } q \in (1, n) \\ 2 & \text{if } q \in [n, \infty) \end{cases}.$$

3.2 Higher integrability in the interior

We now consider the standard results of higher integrability in the interior, which are fundamental in the proof of Theorem 3.1. We generalize them to our Q-quasimini-mizers results in less restrictive conditions; namely no coercivity or growth condition is required.

We state our results here for cubical Q-quasiminimizers and these remain valid for spherical Q-quasiminimizers. We begin with Caccioppoli's inequality which goes back to Boyarskiĭ (1955, 1957) (see also Giusti (2003)).

Theorem 3.4 (**Caccioppoli's inequality in interior**). *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a cubical Q-quasiminimizer of $\int_\Omega (|Dv|^p + 1)\,\mathrm{d}x$, then there exists $c = c(p,Q)$ such that*

$$\int_{C_r} |Du|^p \,\mathrm{d}x \le c\left\{\frac{1}{(R-r)^p}\int_{C_R} |u-u_R|^p \,\mathrm{d}x + |C_R|\right\} \tag{8}$$

and hence

$$⨍_{C_r} |Du|^p \,\mathrm{d}x \le c\left\{\frac{1}{(R-r)^p}⨍_{C_R} |u-u_R|^p \,\mathrm{d}x + 1\right\} \tag{9}$$

for all $x \in \Omega$, $0 < r < R$ and $C_R = C(x,R) \Subset \Omega$ where $|C_R| = \operatorname{meas} C_R$.

Proof. Let C_R be a cube strictly contained in Ω and let $\eta \in C_0^\infty(C_R;\mathbb{R}^N)$ be such that for $r \le s < t \le R$

$$\begin{cases} 0 \le \eta \le 1, \\ \eta = 1 \text{ in } C_s, \\ \eta = 0 \text{ in } C_R \setminus C_t, \\ |D\eta| \le \frac{1}{t-s}. \end{cases}$$

Let

$$\varphi = \eta(u - u_R)$$

where $u_R = ⨍_{C_R} u \,\mathrm{d}x$ and hence $\varphi \in W_0^{1,p}(C_t;\mathbb{R}^N)$. Let

$$v = u - \eta(u - u_R).$$

Then

$$|Dv|^p \le 2^{p-1}(1-\eta)^p|Du|^p + \frac{2^{p-1}}{(t-s)^p}|u-u_R|^p. \tag{10}$$

Since u is a cubical Q-quasiminimizer of the functional $\int_\Omega (|Dv|^p + 1)\,\mathrm{d}x$, we have

$$\int_{C_t} \big(|Du|^p + 1\big)\,\mathrm{d}x \le Q\int_{C_t} \big(|Dv|^p + 1\big)\,\mathrm{d}x$$

which can be rearranged and estimated using (10) to obtain

$$\int_{C_t} |Du|^p \,\mathrm{d}x \le 2^{p-1}Q\int_{C_t\setminus C_s} |Du|^p \,\mathrm{d}x + \frac{2^{p-1}Q}{(t-s)^p}\int_{C_t} |u-u_R|^p \,\mathrm{d}x + (Q-1)|C_t|.$$

Using Widman's hole filling technique (Widman (1971)) results in

$$\int_{C_s} |Du|^p \, dx \leq c \int_{C_t} |Du|^p \, dx + \frac{1}{(t-s)^p} \int_{C_R} |u - u_R|^p \, dx + |C_R|,$$

where $\frac{Q-1}{2^{p-1}Q} \in (0,1)$ and $|C_t| \leq |C_R|$. We now use the iteration lemma to conclude that

$$\int_{C_r} |Du|^p \, dx \leq \tilde{c}(p,c) \left(\frac{1}{(R-r)^p} \int_{C_R} |u - u_R|^p \, dx + |C_R| \right)$$

which then gives us (9) when both sides are divided by $|C_R|$ since clearly $\tilde{c}(p,c) = c(p,Q)$. $\square$

The iteration lemma that was used in the above is stated as follows:

Lemma 3.2 (**Iteration lemma**, Giaquinta and Giusti (1982))**.** *Let $Z(s)$ be a bounded non-negative function in the interval $[r,R]$. Suppose that for all numbers s,t such that $r \leq s < t \leq R$, we have*

$$Z(s) \leq \vartheta Z(t) + A + \frac{B}{(t-s)^\alpha},$$

where $A, B \geq 0$, $\alpha > 0$ and $\vartheta \in [0,1)$ are constants. Then

$$Z(r) \leq c(\alpha, \vartheta) \left(A + \frac{B}{(R-r)^\alpha} \right).$$

The Caccioppoli's inequality is also known as inhomogeneous reverse Poincaré's inequality on increasing support. By Poincaré-Sobolev's inequality, (9) becomes

$$\fint_{C_r} |Du|^p \, dx \leq c \left(\fint_{C_R} |Du|^{\frac{np}{n+p}} \, dx \right)^{\frac{n+p}{n}} + c$$

for all $C_R \subset \Omega$ where $c = c(n,p,Q,r,R)$.

Writing $F := |Du|^{\frac{np}{n+p}}$ and $q = \frac{n+p}{n} = 1 + \frac{p}{n} > 1$, we have the inhomogeneous reverse Hölder's inequality on increasing support

$$\fint_{C_r} F^q \, dx \leq c \left(\fint_{C_R} F \, dx \right)^q + c.$$

By means of Gehring's lemma (Gehring (1973)), the inequality above has a self-improving property:

Lemma 3.3 (**Gehring's lemma**). *Let C_0 be a fixed cube and let $F, G \in L^q(C)$ be non-negative functions. If for some $q > 1$*

$$\fint_{C_R} F^q \,\mathrm{d}x \le c \left(\fint_{C_{2R}} F \,\mathrm{d}x \right)^q + \fint_{C_{2R}} G^q \,\mathrm{d}x + c$$

for each cube C_R with $C_{2R} \subset C$, then there exists $\varepsilon = \varepsilon(c, q, n) > 0$ and $\tilde{c} = \tilde{c}(c, q, n) > 0$ such that

$$\left(\fint_{C_R} F^{\tilde{q}} \,\mathrm{d}x \right)^{\frac{1}{\tilde{q}}} \le \tilde{c} \left\{ \left(\fint_{C_{2R}} F^q \,\mathrm{d}x \right)^{\frac{1}{q}} + \left(\fint_{C_{2R}} G^{\tilde{q}} \,\mathrm{d}x \right)^{\frac{1}{\tilde{q}}} + 1 \right\}$$

for all $\tilde{q} \in [q, q + \varepsilon]$.

This eventually leads to the higher integrability of the gradient of a cubical Q-quasiminimizer of the inhomogeneous p-Dirichlet integral.

Theorem 3.5 (Giusti (2003)). *Let Ω be a bounded Lipschitz domain and let $g \in W^{1,\tilde{p}}(\Omega; \mathbb{R}^N)$ for some $\tilde{p} > p$. If $u \in W_g^{1,p}(\Omega; \mathbb{R}^N)$ is a cubical Q-quasiminimizer of the inhomogeneous p-Dirichlet integral $\int_\Omega (|Dv|^p + 1) \,\mathrm{d}x$, then there exists $\delta \in (0, \tilde{p} - p)$ such that $u \in W^{1,p+\delta}(\Omega; \mathbb{R}^N)$.*

3.3 Proof of Theorem 3.1

The proof of the theorem will involve Caccioppoli's inequality, Poincaré-Sobolév's inequality and iteration lemma.

Proof of Theorem 3.1. Let B_0 be a ball with $\Omega \Subset \frac{1}{2} B_0$. Fix $r > 0$ and let B_r be a ball of radius r with $B_{2r} \subset B_0$. We will divide the proof into two cases.

Case 1: Let $B_{2r} \subset \Omega$. By Caccioppoli's estimate (Theorem 3.4), we have

$$\fint_{B_r} |Du|^p \,\mathrm{d}x \le \frac{c}{r^p} \fint_{B_{2r}} |u - u_{2r}|^p \,\mathrm{d}x + c \tag{11}$$

where constant $c = c(p, Q) > 0$ and $u_{2r} = \fint_{B_{2r}} u \,\mathrm{d}x$. If we let $p_0 = \max \left\{ \frac{np}{n+p}, 1 \right\}$, then we have $p \in (p_0, p_0^*)$ where p_0^* is the Sobolev conjugate of p_0. By (11) and Poincaré-Sobolev's inequality, we obtain

$$\fint_{B_r} |Du|^p \,\mathrm{d}x \le c \left(\fint_{B_{2r}} |Du|^{p(1-\varepsilon)} \,\mathrm{d}x \right)^{\frac{1}{1-\varepsilon}} + c \tag{12}$$

whenever $0 < \varepsilon \leq \min\left\{\frac{p}{n+p}, p-1\right\}$ and $c = c(n, N, p, Q) > 0$.

Case 2: Let $B_{2r} \backslash \Omega \neq \emptyset$. First, we choose a cut-off function $\eta \in C_0^\infty(B_{2r}; \mathbb{R}^N)$ such that for $r \leq s < t \leq 2r$

$$\begin{cases} 0 \leq \eta \leq 1, \\ \eta = 1 \text{ in } B_s, \\ \eta = 0 \text{ in } B_{2r} \setminus B_t, \\ |D\eta| \leq \frac{2}{t-s}. \end{cases}$$

Then $\eta(u-g) \in W_0^{1,p}(B_t \cap \Omega; \mathbb{R}^N)$ and the Q-quasiminimality of u gives us

$$\int_{B_t \cap \Omega} (|Du|^p + 1)\, \mathrm{d}x \leq Q \int_{B_t \cap \Omega} (|Dv|^p + 1)\, \mathrm{d}x$$

where $v = u - \eta(u-g)$. We perform simple algebra and using the fact that $Q|B_t \cap \Omega| - |B_s \cap \Omega| < Q|B_t \cap \Omega|$ to obtain

$$\begin{aligned} \int_{B_s \cap \Omega} |Du|^p \,\mathrm{d}x \leq\; & c \int_{B_t \cap \Omega} (1-\eta)^p |Du|^p \,\mathrm{d}x \\ & + c \int_{B_t \cap \Omega} |u-g|^p |D\eta|^p \,\mathrm{d}x \\ & + c \int_{B_t \cap \Omega} \big((1-\eta)^p + 1\big) |Dg|^p \,\mathrm{d}x \\ & + c|B_t \cap \Omega|. \end{aligned}$$

Applying the assumptions of the cut-off function η yield

$$\begin{aligned} \int_{B_s \cap \Omega} |Du|^p \,\mathrm{d}x \leq\; & c \int_{(B_t \backslash B_s) \cap \Omega} |Du|^p \,\mathrm{d}x \\ & + \frac{c}{(t-s)^p} \int_{B_t \cap \Omega} |u-g|^p \,\mathrm{d}x \\ & + c \int_{B_t \cap \Omega} |Dg|^p \,\mathrm{d}x + c|B_t \cap \Omega|. \end{aligned}$$

By Widman's hole filling technique, we have

$$\begin{aligned} \int_{B_s \cap \Omega} |Du|^p \,\mathrm{d}x \leq\; & \tilde{c} \int_{B_t \cap \Omega} |Du|^p \,\mathrm{d}x \\ & + \frac{\tilde{c}}{(t-s)^p} \int_{B_{2r} \cap \Omega} |u-g|^p \,\mathrm{d}x \\ & + \tilde{c} \int_{B_{2r} \cap \Omega} |Dg|^p \,\mathrm{d}x + \tilde{c}|B_{2r} \cap \Omega|, \end{aligned}$$

where $\tilde{c} = \frac{c}{c+1} \in (0,1)$. Now, by the iteration lemma, we have

$$\int_{B_r\cap\Omega} |Du|^p \, \mathrm{d}x \le \frac{c}{r^p} \int_{B_{2r}\cap\Omega} |u-g|^p \, \mathrm{d}x + c\int_{B_{2r}\cap\Omega} |Dg|^p \, \mathrm{d}x + c|B_{2r}\cap\Omega|. \tag{13}$$

We define $u-g=0$ in $\mathbb{R}^n \setminus \overline{\Omega}$ and therefore $u-g=0$ in Ω^c except for a set of p-capacity zero. We may now apply Lemma 3.1 to estimate $\frac{c}{r^p}\int_{B_{2r}\cap\Omega}|u-g|^p \, dx$. We first let $q = p(1-\varepsilon)$, where

$$\begin{cases} 0<\varepsilon<\min\left\{\frac{p}{n+p}, p-1\right\} & \text{if } p \le n \\ 0<\varepsilon<\min\left\{\frac{p-n}{p}, \frac{1}{2}\right\} & \text{if } p > n \end{cases}.$$

If

$$\kappa = \begin{cases} \frac{n}{n-q} & \text{if } q<n \\ 2 & \text{if } q>n \end{cases},$$

then $\kappa q \ge p$. Applying Lemma 3.1 and performing simple algebra gives us

$$\begin{aligned} &\left(\frac{c}{r^p} \fint_{B_{2r}} |u-g|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \\ \le &c\left(\frac{r^{n-q}}{\mathrm{cap}_q(N(u-g);B_{4r})} \fint_{B_{2r}\cap\Omega} |Du-Dg|^q \, \mathrm{d}x\right)^{\frac{1}{q}}, \end{aligned} \tag{14}$$

where $N(u-g) = \{x \in B_{2r} \colon u(x) = g(x)\}$, the constants c derived for each inequality are actually different from the previous one and the last inequality is due to $u-g=0$ in Ω^c except for a set of p-capacity zero. If $p>n$, then $q>n$ and hence we use the uniform p-thickness of every non-empty set to obtain

$$\begin{aligned} \mathrm{cap}_q(N(u-g);B_{4r}) \ge&\ \mathrm{cap}_q(B_{2r}\setminus\Omega;B_{4r}) \\ \ge&\ c_0\,\mathrm{cap}_q(B_{2r};B_{4r}) \\ \ge&\ cr^{n-q}. \end{aligned} \tag{15}$$

If $p \le n$, then it follows from Proposition 3.2 that Ω^c is uniformly q-thick if $\varepsilon \in \big(0, \varepsilon_0(n,p,c_0)\big)$. Similarly, for ε small enough, we have (15). Putting the estimates together, we finally have

$$\left(\frac{1}{r^n}\int_{B_r\cap\Omega} |Du|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \le c + c\left(\frac{1}{r^n}\int_{B_{2r}\cap\Omega} |Du|^q \, \mathrm{d}x\right)^{\frac{1}{q}} + c\left(\frac{1}{r^n}\int_{B_{2r}\cap\Omega} |Dg|^p \, \mathrm{d}x\right)^{\frac{1}{p}}. \tag{16}$$

Let

$$f(x) = \begin{cases} |Du(x)|^{p(1-\varepsilon)} & \text{if } x \in \Omega \\ 0 & \text{elsewhere} \end{cases}$$

and

$$h(x) = \begin{cases} |Dg(x)|^{p(1-\varepsilon)} & \text{if } x \in \Omega \\ 0 & \text{elsewhere} \end{cases}.$$

Further let $s = \frac{1}{1-\varepsilon} > 1$, where $\varepsilon \in (0,1)$ is so small that both (12) and (16) hold. Then $f, h \in L^s(B_0; \mathbb{R}^N)$ and we obtain the following inhomogeneous reverse Hölder's inequality on increasing supports

$$\fint_{B_r} f^s \,\mathrm{d}x \leq c \left(\fint_{B_{2r}} f \,\mathrm{d}x \right)^s + c \fint_{B_{2r}} h^s \,\mathrm{d}x + c$$

whenever $B_{2r} \subset B_0$. It then follows from Gehring's lemma that there exist $\tilde{\varepsilon} = \tilde{\varepsilon}(c,s,n) > 0$ and $\tilde{c} = \tilde{c}(c,s,n) > 0$ such that

$$\left(\fint_{B_r} f^t \,\mathrm{d}x \right)^{\frac{1}{t}} \leq \tilde{c} \left(\fint_{B_{2r}} f^s \,\mathrm{d}x \right)^{\frac{1}{s}} + \tilde{c} \left(\fint_{B_{2r}} h^t \,\mathrm{d}x \right)^{\frac{1}{t}} + \tilde{c} \tag{17}$$

for all $t \in [s, s+\tilde{\varepsilon}]$. Since Ω is bounded, we can choose a finite number of balls $B(x_j, r_j)$, $j = 1, \ldots, J$ such that

$$B(x_j, 2r_j) \subset B_0$$

and

$$\Omega \subset \bigcup_{j=1}^{J} B(x_j, r_j).$$

This leads to the inequality

$$\left(\int_\Omega f^t \,\mathrm{d}x \right)^{\frac{1}{t}} \leq c \left(\int_\Omega f^s \,\mathrm{d}x \right)^{\frac{1}{s}} + c \left(\int_\Omega h^t \,\mathrm{d}x \right)^{\frac{1}{t}} + c \tag{18}$$

where the constant c has been adjusted and therefore

$$f \in L^{s+\delta'}(\Omega; \mathbb{R}^N)$$

for some $\delta' = \delta'(n, N, p, Q, c_0) \in [0, \tilde{\varepsilon}]$ provided that

$$g \in L^{s+\delta'}(\Omega; \mathbb{R}^N).$$

We substitute $|Du|$ and $|Dg|$ into (18) to get

$$\left(\int_\Omega |Du|^{p(1+\delta'(1-\varepsilon))} \,\mathrm{d}x \right)^{\frac{1}{p(1+\delta'(1-\varepsilon))}}$$

$$\leq c \left(\int_\Omega |Du|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + \left(\int_\Omega |Dg|^{p\left(1+\delta'(1-\varepsilon)\right)} \,\mathrm{d}x \right)^{\frac{1}{p\left(1+\delta'(1-\varepsilon)\right)}} + c$$

Clearly $1+\delta'(1-\varepsilon) > 1$, so we may write $p\big(1+\delta'(1-\varepsilon)\big)$ as $p+\delta$ and therefore

$$\left(\int_\Omega |Du|^{p+\delta}\,\mathrm{d}x\right)^{\frac{1}{p+\delta}} \le c\left(\int_\Omega |Du|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |Dg|^{p+\delta}\,\mathrm{d}x\right)^{\frac{1}{p+\delta}} + c$$

provided that $\delta \in (0,\delta_0)$ where $\delta_0 = \delta_0(n,N,p,Q,c_0) \in (0,\tilde{p}-p]$ and $g \in W^{1,\tilde{p}}(\Omega;\mathbb{R}^N)$ for some $\tilde{p} > p$. Finally, it follows from Sobolev's inequality that $u \in W^{1,p+\delta}(\Omega;\mathbb{R}^N)$. □

4. Conclusion

We have discussed the two most attractive features of Q-quasiminimizer; namely its generality and regularity properties. Further strengthening its wide range of applications, we have shown how Q-quasiminimizers appear naturally in variational problems – if a variational problem has a minimizing sequence, then there exists another minimizing sequence for the variational problem and moreover it consists of Q-quasiminimizers for the inhomogeneous p-Dirichlet integral. The gradients of these Q-quasiminimizers are not only higher integrability in the interior, but also in the whole set of Ω provided that the complement of Ω is uniformly p-thick.

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